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ON THE PHENOMENON OF DECOHESION IN PERFECT PLASTICITY

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Abstract—In certain problems of perfectly elastic–plastic bodies the strains may increase infinitely causing the discontinuities which are not permissible from the viewpoint of continuous media. Then the phenomenon of decohesion may occur; the corresponding loading is called "decohesive carrying capacity". It results in a redistribution of internal forces and either leads to immediate exhaustion of the limit carrying capacity of the structure or the further work under increasing loading parameters is still possible. Two corresponding examples are discussed; in any case the decohesive carrying capacity determines the real strength of the structure from the engineering point of view.

1. INTRODUCTION

THE theory of perfectly elastic–plastic bodies usually describes two types of load-carrying capacity of the considered structure: the elastic carrying capacity (onset of first plastic deformations) and the limit carrying capacity (maximal loading, corresponding to a certain mechanism of plastic collapse). The strains may increase arbitrarily and as a rule no problem of decohesion (fracture) is analyzed. Some attempts have been made to describe such phenomena by separate equations, for example limiting the elongations (Davidenkov and Fridman, cf. Fridman [2]; Pełczyński [8]; Zakrzewski [16]; Labutin [6]).

In certain problems, however, even the assumption of arbitrarily large strains is not sufficient to reach a mechanism of plastic collapse without discussing the problem of decohesion. Simply the locally infinite increase of strains leads to discontinuities of displacements which are not permissible from the point of view of continuous media.

The permissible discontinuities were discussed by Hill [4, 5], Thomas [12, 13], Prager [10], Freudenthal and Geiringer [1], but, for example, a jump of displacements or velocities normal to the discontinuity surface is never permissible. If such a jump takes place, we have to discuss the decohesion (at least for positive longitudinal strains) and its further influence on the behavior of the considered structure. Such a local decohesion results always in redistribution of the internal forces. In certain cases this redistribution causes immediate collapse of the structure, in other cases the mechanism of plastic collapse is reached at larger values of the loadings. In both alternatives, however, the decohesion seems to be inadmissible from the point of view of engineering applications. The corresponding loading (or intensity of loadings) will be called here the "decohesive carrying capacity", and if such a loading exists, it will be suggested as an estimation of real carrying capacity of a perfectly elastic–plastic structure.

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2. DECOHESION RESULTING IN IMMEDIATE COLLAPSE: A SIMPLE BAR SYSTEM

2.1 *Perfect plasticity*

Consider a system of three deformable vertical bars and a rigid horizontal beam. The bars are either tapered [Fig. 1(a)] or their own weight is taken into account [Fig. 1(b)]. The material of the bars is assumed to be perfectly elastic–plastic, with Young's modulus E and yield stress σ_0. All the connections are perfect pin joints.

We shall discuss here the second system, Fig. 1(b), which seems to be more simple. In the elastic range the reactive forces are equal to

$$R_1 = \frac{P}{12}+\gamma Al, \qquad R_2 = \frac{P}{3}+\gamma Al, \qquad R_3 = \tfrac{7}{12}P+\gamma Al \tag{2.1}$$

where γ denotes the specific weight of the material of the bars, A—the cross-sectional area, assumed here as constant. The weight of the rigid beam is here disregarded, but it would introduce no change to the discussion.

The largest stress occurs at the fixed cross-section under the reactive force R_3 and equals R_3/A. Thus the equation

$$\frac{7}{12}\frac{P}{A}+\gamma l = \sigma_0 \tag{2.2}$$

describes the elastic carrying capacity of the system (we assume no collapse under own weight alone, $\gamma l < \sigma_0$). But no further increase of the force P is here possible without decohesion. As a matter of fact, the yield stress σ_0 in the third bar is reached at the cross-section $x = 0$ only. It may not be reached at any other section, since R_3 cannot exceed the

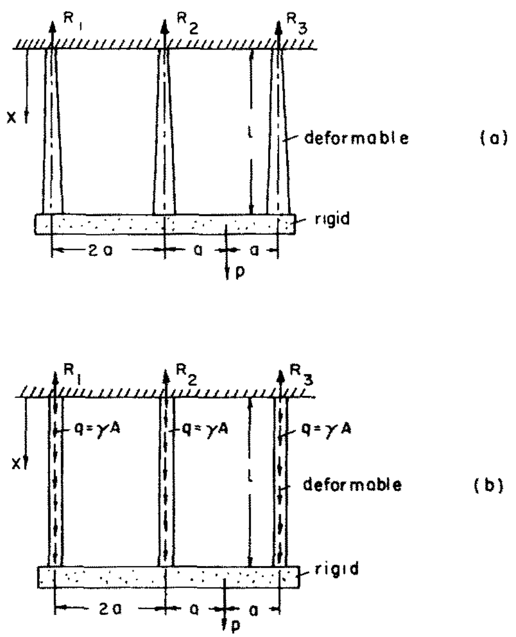

FIG. 1. Simple bar systems subject to decohesion.

value $A\sigma_0$ and the bar remains elastic with the stress distribution

$$\sigma = \frac{R_3}{A} - \gamma x = \sigma_0 - \gamma x. \tag{2.3}$$

Thus the increase of elongation of the bar "3", necessary for further work of the system, results in infinite increase of the strain ε at $x = 0$. But infinite value of du/dx corresponds to a jump in the displacement u, thus with local decohesion. The equation (2.2) describes then the decohesive carrying capacity of the system as well.

Consider now the system after decohesion, Fig. 2, with $R_3 = 0$. The system is now statically determinate and the reactive forces are equal:

$$R_1 = -\frac{P}{2}, \qquad R_2 = \tfrac{3}{2}P + 3\gamma Al. \tag{2.4}$$

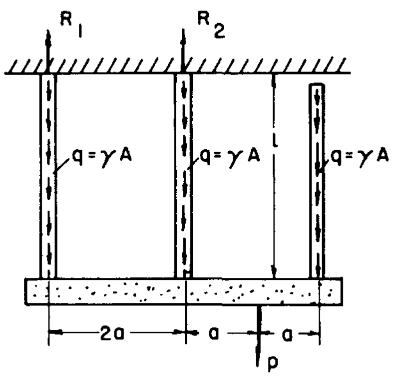

FIG. 2. Bar system after decohesion.

Substituting here the force P, determined by (2.2), namely

$$P = \tfrac{12}{7}A(\sigma_0 - \gamma l) \tag{2.5}$$

we obtain the stresses in the bar "2" much exceeding the yield stress, thus the system will collapse immediately. The limit carrying capacity must be here regarded as described by (2.5) too.

An interesting discussion is connected with the limiting case $\gamma \to 0$. Equation (2.5) yields then $P = (\frac{12}{7})A\sigma_0$. But the limit carrying capacity of the weightless system is $\bar{P} = 2A\sigma_0$ (with the mechanism of plastic collapse corresponding to the rotation about the hinge "1"). Thus the infinitely small own weight (or infinitely small tapering) leads here to a jump in the value of the limit carrying capacity. This jump may not be weakened in the framework of the theory of perfect plasticity, for example by taking the decrease of the cross-sectional area into consideration (on the contrary, this would lead to more sudden decohesion); however, it may be reduced for certain types of other stress–strain diagrams.

2.2 *Deviations from perfect plasticity*

Strain hardening, occurring after the yield point has been reached, usually does not allow to discuss the plastic carrying capacity of the structure, unless a separate criterion of rupture is specified or geometrical changes are taken into account (influence of deflections on the bending moment, decrease of the cross-sectional area etc.). But there exist other deviations from perfect plasticity, allowing for the discussion of limit carrying capacity, namely occurring below the yield stress (Fig. 3). Such diagrams, called sometimes "strain-hardening" without sufficient justification, will be named here "asymptotically perfect plasticity". The limit carrying capacity, found on the basis of such diagrams, usually coincides with that determined for a perfectly plastic material.

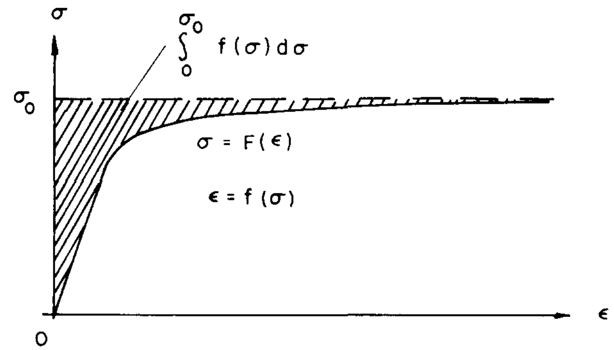

FIG. 3. Asymptotically perfect plasticity.

We are now going to analyze how the asymptotically perfect plasticity may result in the discussion of decohesion, particularly for the system of bars under consideration.

The main problem is whether the elongation of a bar (in our case—of the bar "3") may increase infinitely or not. The infinite increase of Δl is necessary to avoid decohesion and to obtain in the limiting case $\gamma \to 0$ the limit carrying capacity as predicted by the theory for weightless bars, $\bar{\bar{P}} = 2A\sigma_0$. Elongation of the bar is equal to

$$\Delta l = \int_0^l \varepsilon \, dx = \int_0^l f(\sigma) \, dx = \int_{\sigma(0)}^{\sigma(l)} \frac{f(\sigma)}{d\sigma/dx} \, d\sigma, \tag{2.6}$$

where $\varepsilon = f(\sigma)$ is the equation of the stress–strain curve, Fig. 3. In our case (2.3) yields $d\sigma/dx = -\gamma = \text{const.}$, thus the discussion is reduced to the problem, whether the improper integral

$$I = -\int_{\sigma(0)}^{\sigma(l)} f(\sigma) \, d\sigma = \int_{\sigma(l)}^{\sigma_0} f(\sigma) \, d\sigma \tag{2.7}$$

is convergent or not, since we have to substitute $\sigma(0) = \sigma_0$ as the maximal admissible value of the stress. The integral (2.7) may be interpreted as a part of the complementary specific energy, Fig. 3.

Take, for example, two most commonly used equations, describing asymptotically perfect plasticity: that proposed by Prager [9]:

$$\sigma = \sigma_0 \tanh\left(\frac{E\varepsilon}{\sigma_0}\right), \quad \text{thus } \varepsilon = f(\sigma) = \frac{\sigma_0}{E} \tanh_{-1}\left(\frac{\sigma}{\sigma_0}\right), \tag{2.8}$$

or proposed by Ylinen [15]

$$\varepsilon = \frac{1}{E}\left[c\sigma - (1-c)\sigma_0 \ln\left(1 - \frac{\sigma}{\sigma_0}\right)\right], \tag{2.9}$$

where c is a dimensionless material constant. In both cases the integral I, (2.7), is convergent, the elongation (2.6) is limited and the decohesion must occur before reaching the mechanism of plastic collapse.

To avoid decohesion we have to consider the stress–strain curves approaching less rapidly the yield stress σ_0. For example, analyzing the functions of the type

$$\varepsilon = f(\sigma) = \frac{\sigma}{E(1-\sigma/\sigma_0)^n} \tag{2.10}$$

we find that for $n \geq 1$ the integral (2.7) is divergent, the elongations may increase infinitely and the mechanism of the plastic collapse may be reached without decohesion.

For the system with non-prismatic bars, Fig. 1(a), the discussion may be a bit more complicated, since the derivative $d\sigma/dx$ is not necessarily finite (larger than zero). If it is finite at $x = 0$, as for the cones, pyramids etc., the discussion remains as before. But if $dA/dx = 0$ at $x = 0$, which causes $d\sigma/dx = 0$ at this point, the class of the functions $\varepsilon = f(\sigma)$ resulting in plastic yielding without decohesion is much wider, since the integral (2.6) is more often divergent. This integral may no more be interpreted here as the complementary specific energy. Even for perfect plasticity the integral (2.6) may be in this case divergent and the system may reach its limit carrying capacity without decohesion.

3. DECOHESION RESULTING IN THE REDISTRIBUTION OF STRESSES WITHOUT IMMEDIATE COLLAPSE: A SHEET WITH RIGID INCLUSION

3.1 *Elastic carrying capacity*

Consider an infinite sheet with rigid circular inclusion of the radius a, Fig. 4. The sheet (plate), made of perfectly elastic–plastic material, is permanently connected with the inclusion and is subject to uniform biaxial tension by the traction p at infinity. The usual assumption of plane stress in the sheet will be made. Polar coordinates $r-\theta$ will be used.

For small values of the traction p the whole sheet will be elastic and the general expressions for the stresses and radial displacement are

$$\sigma_r = A + \frac{B}{r^2} \qquad \sigma_\theta = A - \frac{B}{r^2}$$
$$u = \frac{1}{E}\left[(1-\nu)Ar - (1+\nu)\frac{B}{r}\right]. \tag{3.1}$$

The boundary conditions

$$u(a) = 0 \qquad \sigma_r(\infty) = \mathrm{p} \tag{3.2}$$

yield

$$A = p \qquad B = \frac{1-\nu}{1+\nu}pa^2. \tag{3.3}$$

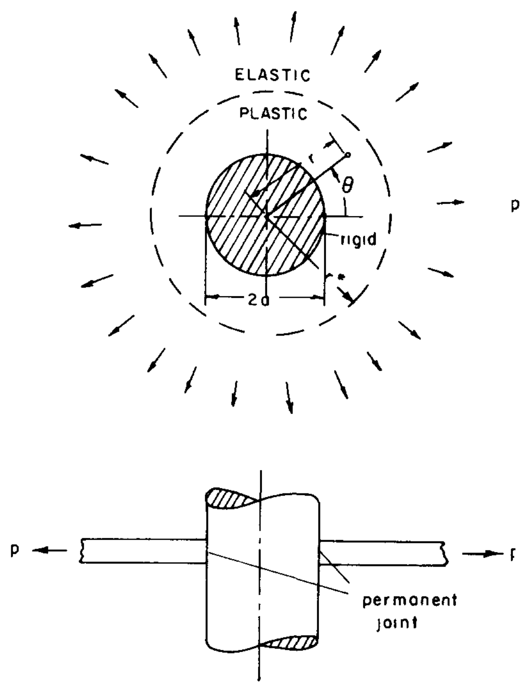

FIG. 4. Infinite sheet with rigid circular inclusion.

Assuming the Huber–Mises–Hencky yield condition we describe the stress intensity σ_i (reduced stress according to the HMH hypothesis) by the formula

$$\sigma_i = \sqrt{(\sigma_r^2+\sigma_\theta^2-\sigma_r\sigma_\theta)} = \sqrt{\left(A^2+3\frac{B^2}{r^4}\right)}; \tag{3.4}$$

the upper bound of σ_i is reached for $r = a$, thus the equation of the elastic carrying capacity (onset of the first plastic strains) is determined by

$$A^2+3\frac{B^2}{a^4} = \sigma_0^2, \tag{3.5}$$

so that the corresponding loading $\bar{p}$ amounts

$$\bar{p} = \frac{1+v}{2\sqrt{(1-v+v^2)}}\sigma_0. \tag{3.6}$$

3.2 *The Prandtl–Reuss theory*

For larger values of p the plastic zone $a < r < r^*$ will appear (with the exception of the case of an incompressible material, $v = 1/2$, which will be discussed later). The distribution of stresses in the plastic zone is statically determinate (up to a certain constant) and may be determined regardless of any particular theory of plasticity. Use of the Nadai–Sokolovsky parametrization of the yield condition

$$\sigma_r = \frac{2}{\sqrt{3}}\sigma_0 \sin\zeta$$
$$\sigma_\theta = \frac{2}{\sqrt{3}}\sigma_0 \sin\left(\zeta+\frac{\pi}{3}\right) \tag{3.7}$$

where ζ is a parameter, the distribution of which $\zeta = \zeta(r)$ should be found. Both stresses σ_r and σ_θ are here nonnegative and it turns out that $\sigma_r > \sigma_\theta$, thus $\pi/3 \leq \zeta \leq 2\pi/3$. Substituting (3.7) into the condition of internal equilibrium in polar coordinates, we obtain the equation

$$\cos\zeta\,\zeta' + \frac{1}{r}\left[\sin\zeta - \sin\left(\zeta+\frac{\pi}{3}\right)\right] = 0 \tag{3.8}$$

which may be solved with respect to r, thus determining the function inverse with respect to $\zeta = \zeta(r)$:

$$r = \frac{C_1 \exp\{[\sqrt{(3)}/2]\zeta\}}{\sqrt{[\sin(\zeta-\pi/3)]}} \tag{3.9}$$

C_1 being the constant of integration. The boundary condition for the plastic zone, at $r = a$, refers to the displacements (or velocities) and not to the stresses, thus the constant C_1 cannot be evaluated without considering the distribution of displacements.

To find this distribution in the plastic zone let us use, at first, the Prandtl–Reuss theory, which is best justified for perfectly elastic–plastic bodies. The general equations

$$\dot\varepsilon_{ij} - \delta_{ij}\dot\varepsilon_m = \frac{1}{2G}(\dot\sigma_{ij} - \delta_{ij}\dot\sigma_m) + \lambda(\sigma_{ij} - \delta_{ij}\sigma_m)$$
$$\varepsilon_m = \frac{1}{3K}\sigma_m \tag{3.10}$$

(where dots denote the differentiation with respect to the time t, $\varepsilon_m = \varepsilon_{\alpha\alpha}/3$ and $\sigma_m = \sigma_{\alpha\alpha}/3$ stand for the mean strain and mean stress, respectively, and K is the bulk modulus), may be written here as follows:

$$\dot\varepsilon_r = \frac{1}{3K}\dot\sigma_m + \frac{1}{2G}(\dot\sigma_r - \dot\sigma_m) + \lambda(\sigma_r - \sigma_m),$$
$$\dot\varepsilon_\theta = \frac{1}{3K}\dot\sigma_m + \frac{1}{2G}(\dot\sigma_\theta - \dot\sigma_m) + \lambda(\sigma_\theta - \sigma_m). \tag{3.11}$$

The elimination of the unknown function λ and the substitution of $\sigma_m = (\sigma_r + \sigma_\theta)/3$ yields the equation

$$\left[\dot\varepsilon_r - \frac{1}{9K}(\dot\sigma_r + \dot\sigma_\theta) - \frac{1}{6G}(2\dot\sigma_r - \dot\sigma_\theta)\right](2\sigma_\theta - \sigma_r)$$
$$= \left[\dot\varepsilon_\theta - \frac{1}{9K}(\dot\sigma_r + \dot\sigma_\theta) - \frac{1}{6G}(2\dot\sigma_\theta - \dot\sigma_r)\right](2\sigma_r - \sigma_\theta). \tag{3.12}$$

The stresses σ_r and σ_θ are known and we are looking for ε_r and ε_θ. They may be both expressed in terms of the radial displacement u or the compatibility equation may be used, expressing ε_r in terms of ε_θ. It turns out that the equation with respect to ε_θ is more simple. Thus we substitute

$$\varepsilon_r = \varepsilon_\theta + r\frac{\partial \varepsilon_\theta}{\partial r} \tag{3.13}$$

and making use of the expressions for stresses, (3.7) we arrive after some simple rearrangements at the following partial differential equation with respect to ε_θ:

$$r\frac{\partial \dot{\varepsilon}_\theta}{\partial r} + \sqrt{3}\frac{\cos(\zeta+\pi/6)}{\cos\zeta}\dot{\varepsilon}_\theta = \frac{2\sigma_0\dot{\zeta}}{\sqrt{3}\cos\zeta}\left[\frac{1}{4G}+\frac{1}{3K}\cos^2\left(\zeta+\frac{\pi}{6}\right)\right]. \tag{3.14}$$

The function ζ must be considered now as a function of two variables, $\zeta = \zeta(r, t)$, determined by the inverse relation (3.9), in which C_1 depends on the time t, $C_1 = f_1(t)$. The symbol $\dot{\zeta}$ denotes the derivative $\partial\zeta/\partial t$. If we present (3.9) in the form $F(\zeta, r, t) = 0$, then

$$\dot{\zeta} = -\frac{\partial F/\partial t}{\partial F/\partial \zeta} = -\frac{f_1'(t)\cos(\zeta+\pi/6)}{f_1(t)\cos\zeta}. \tag{3.15}$$

The equation (3.14) may be considered as an ordinary linear differential equation with respect to $\dot{\varepsilon}_\theta$ with time t as a parameter. Its general solution may be expressed in terms of the parameter ζ as follows:

$$\begin{aligned}\dot{\varepsilon}_\theta = &-\frac{2\sigma_0}{\sqrt{3}}\frac{f_1'(t)}{f_1(t)}\exp[-\sqrt{(3)}\zeta]\int\left[\frac{1}{4G}+\frac{1}{3K}\cos^2\left(\zeta+\frac{\pi}{6}\right)\right]\frac{\exp[\sqrt{(3)}\zeta]}{\cos\zeta}\,d\zeta\\ &+f_2(t)\exp[-\sqrt{(3)}\zeta].\end{aligned} \tag{3.16}$$

This solution is rather involved and unsuitable for our further analysis; the boundary conditions at the elastic–plastic interface are here complicated too. The method of successive approximations, applied by Tuba [14] (for the case of linear strain-hardening) seems to be unsuitable for our purposes. Thus we are going to apply a simpler theory of plasticity. In any case it is seen that $\zeta \to \pi/2$ leads to a singularity, namely to non-permissible discontinuity, which results in decohesion at the corresponding radius.

3.3 *The Hencky–Ilyushin theory*

Two theories of plasticity are simpler than the Prandtl–Reuss theory. The equations of Levy–Mises were used in a similar problem of enlargement of a circular hole in an infinite sheet by Taylor [11] and Hill [3]. However, these equations correspond rather to an incompressible rigid-plastic body and an attempt by Taylor to combine his results for the plastic zone with the known solution for the elastic zone seems to be somewhat sophisticated. In any case the discussion of the influence of compressibility of the material on the decohesive carrying capacity, very important in the considered case, would be impossible.

To obtain more simple results we are going to use here the theory of Hencky–Ilyushin, which leads to correct results for simple loading (proportionally increasing components of the strain deviator, or of the stress deviator, which is then equivalent). Approximately it may be used also for the loading paths deviating just slightly from simple loading. In our case this deviation may be estimated. Plastic deformations start at $r = a$ with $\sigma_r = 2p/(1+\nu)$, $\sigma_\theta = 2\nu p/(1+\nu)$ and—according to the previous conclusion—end if $\zeta \to \pi/2$, it means $\sigma_r = 2\sigma_0/\sqrt{3}$, $\sigma_\theta = \sigma_0/\sqrt{3}$. Thus the ratio of the components of the stress deviator, $(\sigma_\theta - \sigma_m)/(\sigma_r - \sigma_m)$, varies from $-(1-2\nu)/(2-\nu)$ to 0. For not too small value of the Poisson's ratio ν this variation is insignificant.

Instead of (3.10) we write now

$$\varepsilon_{ij} - \delta_{ij}\varepsilon_m = \varphi(\sigma_{ij} - \delta_{ij}\sigma_m), \tag{3.17}$$

and instead of (3.11)

$$\begin{aligned} \varepsilon_r &= \frac{1}{3K}\sigma_m + \varphi(\sigma_r - \sigma_m) \\ \varepsilon_\theta &= \frac{1}{3K}\sigma_m + \varphi(\sigma_\theta - \sigma_m). \end{aligned} \tag{3.18}$$

Elimination of the unknown function φ leads to the equation

$$(2\sigma_\theta - \sigma_r)\varepsilon_r - (2\sigma_r - \sigma_\theta)\varepsilon_\theta = \frac{1}{3K}(\sigma_\theta^2 - \sigma_r^2). \tag{3.19}$$

General solution of this equation with respect to the radial displacement u and the strains ε_r and ε_θ was given in the paper [17]. Here it is more convenient to express the stresses by means of the parameter ζ, (3.7). Using the compatibility condition (3.13) we obtain

$$r\cos\zeta\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}r} + \sqrt{3}\cos\left(\zeta + \frac{\pi}{6}\right)\varepsilon_\theta = \frac{4\sigma_0}{3K\sqrt{3}}\sin\left(\zeta + \frac{\pi}{6}\right)\cos\left(\zeta + \frac{\pi}{6}\right) \tag{3.20}$$

or, making use of (3.8) and eliminating r,

$$\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}\zeta} + \sqrt{(3)}\varepsilon_\theta = \frac{2\sigma_0}{3K\sqrt{3}}\sin\left(\zeta + \frac{\pi}{6}\right). \tag{3.21}$$

The solution of this linear ordinary differential equation may be written in the form

$$\varepsilon_\theta = \frac{\sigma_0}{3K\sqrt{3}}\sin\zeta + C_2\exp[-\sqrt{(3)}\zeta]. \tag{3.22}$$

The symbol C_2 stands here for the constant of integration, corresponding to the function $f_2(t)$ in the general solution (3.16) for $\dot{\varepsilon}_\theta$.

Thus, making use of (3.9), we express the radial displacement u in terms of the parameter ζ as follows:

$$u = \frac{C_1\sigma_0\sin\zeta\exp\{[\sqrt{(3)}/2]\zeta\}}{3K\sqrt{3}\sqrt{[\sin(\zeta - \pi/3)]}} + C_1C_2\frac{\exp\{-[\sqrt{(3)}/2]\zeta\}}{\sqrt{[\sin(\zeta - \pi/3)]}}. \tag{3.23}$$

The radial strain ε_r equals

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\mathrm{d}u}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}r} \tag{3.24}$$

and having performed the differentiations and several simple rearrangements of the trigonometric functions, we obtain finally

$$\varepsilon_r = \frac{\sigma_0}{3K\sqrt{3}}\sin\left(\zeta+\frac{\pi}{3}\right)+\frac{C_2}{\cos\zeta}\sin\left(\zeta-\frac{\pi}{6}\right)\exp[-\sqrt{(3)}\zeta]. \tag{3.25}$$

Now we may combine the general solution for the elastic zone, (3.1) denoted further by superscript (e), with the general solution for the plastic one, (3.7), (3.9), (3.22), (3.23) and (3.25), denoted by (p). The number of constants and of boundary conditions is rather high, since the solution is given in parametrical form. We have to evaluate four constants of integration, A, B, C_1 and C_2, radius separating the zones r^* and the values of the parameter ζ_a and ζ^*, corresponding to a and r^*, thus totally seven unknowns. To this aim we have the following seven boundary conditions:

$$u^{(p)} = 0 \quad \text{and} \quad \zeta = \zeta_a \quad \text{for } r = a, \tag{3.26}$$

$$u^{(p)} = u^{(e)}, \qquad \sigma_r^{(p)} = \sigma_r^{(e)}, \qquad \sigma_i^{(e)} = \sigma_0, \quad \text{and} \quad \zeta = \zeta^* \quad \text{for } r = r^*, \tag{3.27}$$

$$\sigma_r^{(e)} = p \quad \text{for } r = \infty, \tag{3.28}$$

where $\sigma_i^{(e)}$ is determined by (3.4). Solving subsequently these equations we may express six unknowns in terms of the remaining one, ζ_a, as follows

$$\begin{aligned}
A &= p = q\sigma_0 \\
B &= \frac{\sigma_0 a^2}{\sqrt{3}}\sin\left(\zeta_a-\frac{\pi}{3}\right)\exp\left[\sqrt{3}\left(\frac{\pi}{3}+\arccos q-\zeta_a\right)\right] \\
C_1 &= a\sqrt{\left[\sin\left(\zeta_a-\frac{\pi}{3}\right)\right]}\exp\left(-\frac{\sqrt{3}}{2}\zeta_a\right) \\
C_2 &= -\frac{\sigma_0}{3K\sqrt{3}}\sin\zeta_a\exp[\sqrt{(3)}\zeta_a] \\
r^* &= \frac{a}{\sqrt[4]{(1-q^2)}}\sqrt{\left[\sin\left(\zeta_a-\frac{\pi}{3}\right)\right]}\exp\left[\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}+\arccos q-\zeta_a\right)\right] \\
\zeta^* &= \pi-\arcsin\frac{\sqrt{(1-q^2)}+q\sqrt{3}}{2} = \frac{\pi}{3}+\arccos q,
\end{aligned} \tag{3.29}$$

where $q = p/\sigma_0$ denotes the dimensionless traction at infinity. The last unknown, ζ_a, is determined by a transcendental equation, $f(\zeta_a, q, \nu) = 0$. However, this equation may be solved with respect to the Poisson's ratio ν, and an inverse procedure may be used:

$$\nu = \frac{1}{2}-\frac{3\sqrt{(1-q^2)}-q\sqrt{3}}{4\sin\zeta_a}\exp\left[\sqrt{3}\left(\frac{\pi}{3}+\arccos q-\zeta_a\right)\right] \tag{3.30}$$

Figure 5 presents the diagram of the even simpler function $\nu = \nu(\zeta_a, \zeta^*)$, namely

$$\nu = \frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\cos\zeta^*}{\sin\zeta_a}\exp[\sqrt{(3)}(\zeta^*-\zeta_a)] \tag{3.31}$$

and makes it possible to find $\zeta_a = \zeta_a(\zeta^*, \nu)$ in a graphical way. This diagram visualizes also the propagation of the plastic deformations. Plastic zone starts, when $\zeta^* = \zeta_a$ (limiting

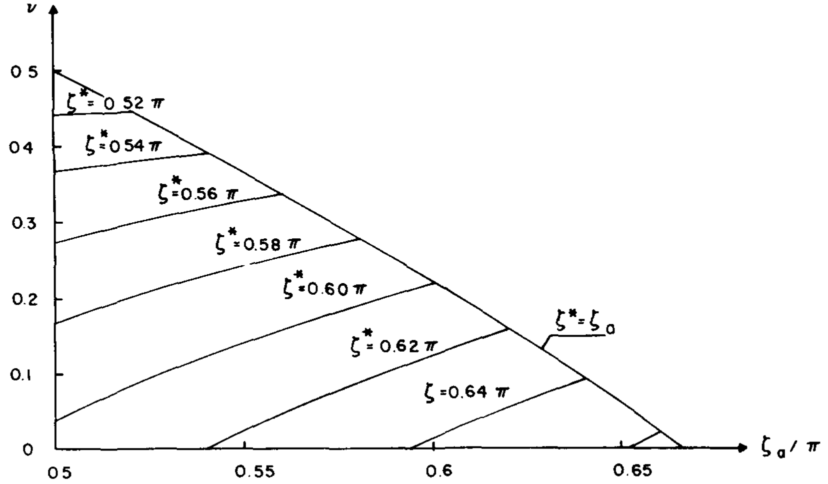

FIG. 5. Relation between Poisson's ratio ν and auxiliary angles ζ_a and ζ^*.

line). With increasing load both ζ_a and ζ^* decrease (motion along ν = const. to the left). This process ends for $\zeta_a = \pi/2$ as it will be shown in Section 3.4.

3.4 *Decohesive carrying capacity*

Substituting (3.29) into (3.25) we determine radial strain ε_r by the formula

$$\varepsilon_r^{(p)} = \frac{\sigma_0}{3K\sqrt{3}}\left\{\sin\left(\zeta+\frac{\pi}{3}\right)-\frac{\sin\zeta_a\sin(\zeta-\pi/6)}{\cos\zeta}\exp[\sqrt{(3)}(\zeta_a-\zeta)]\right\}. \tag{3.32}$$

The maximal value of $\varepsilon_r^{(p)}$ occurs at $r = a$ and equals

$$\varepsilon_{ra}^{(p)} = \frac{\sigma_0}{3K\sqrt{3}}\,\frac{\cos(2\zeta_a-\pi/6)}{\cos\zeta_a}. \tag{3.33}$$

For ζ_a approaching $\pi/2$, $\varepsilon_{ra}^{(p)}$ increases infinitely, which leads to a nonpermissible discontinuity from the point of view of continuous media. This corresponds to decohesion at $r = a$ (the Prandtl–Reuss theory gives qualitatively the same result). The corresponding decohesive carrying capacity, denoted here by $\hat{q}$, may be found from the inverted last formula (3.29)

$$\hat{q} = \cos\left(\zeta^*-\frac{\pi}{3}\right) \tag{3.34}$$

where ζ^* is determined by the diagram 5 for $\zeta_a = \pi/2$ and for the given value of Poisson's ratio ν. Thus $\hat{q}$ depends on ν; this dependence is shown in Fig. 6 together with the plots $\bar{q} = \bar{q}(\nu)$, (3.6) and the limit carrying capacity $\bar{\bar{q}}$, determined below. For an incompressible material, $\nu = \frac{1}{2}$, the plastic zone starts at $\zeta_a = \pi/2$ and the decohesion occurs immediately, $\hat{q} = \bar{q}$.

The decohesion at $r = a$ is here not equivalent to the exhaustion of the limit carrying capacity. It leads, at first, to passive processes (unloading) in the zone $a < r < r^*$. The

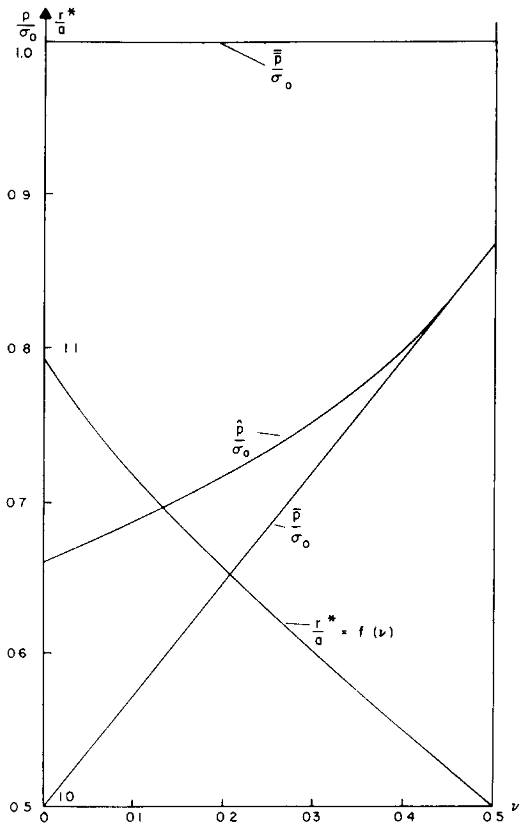

FIG. 6. Elastic, decohesive and limit carrying capacities of the sheet with inclusion.

stress σ_r ar $r = a$ decreases from $(2/\sqrt{3})\sigma_0 \sin \zeta_a$ to 0, but the plate will work further as a plate with a circular hole. In such a plate $0 \leq \sigma_r \leq \sigma_\theta$ and yielding will correspond to ζ from the interval $0 \leq \zeta \leq \pi/3$, different from the interval holding before decohesion. The limit carrying capacity in this case, $\bar{\bar{p}}$, is determined for example by Nadai [7]; it turns out, that $\bar{\bar{p}} = \sigma_0$, it means $\bar{\bar{q}} = 1$. This function is also shown in Fig. 6.

3.5 *Example of stress and strain distribution*

An example of the distribution of stresses and strains at the moment of decohesion (start of decohesion) is presented in Fig. 7. Poisson's ratio is here assumed to be $\nu = 0{\cdot}3$ and the radius r^*, determined by (3.29), equals $r^* = 1.024a$. Thus the decohesion corresponds here to a small but finite plastic zone (in Section 2 decohesion was connected with infinitely small plastic zone). Only in the case of an incompressible body decohesion appears immediately, with the onset of plastic deformations.

4. FINAL REMARKS

The decohesive carrying capacity, if such exists, is from the point of view of engineering applications usually much more important than the limit carrying capacity, even if the last

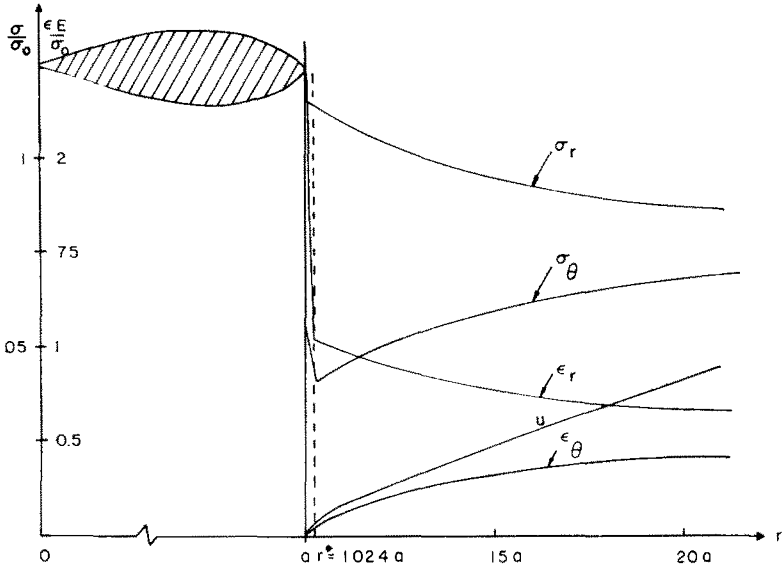

FIG. 7. Distribution of stresses, strains and displacements in the sheet at the moment of decohesion.

is much higher. As a matter of fact, decohesion should almost always be avoided. Take, for example, a rotating annular disk joined permanently with a driving shaft. This example resembles that discussed in Section 3, but is more difficult to analytical treatment. At a certain angular velocity $\hat{\omega}$ decohesion will occur. Although it is not equivalent to the exhaustion of limit carrying capacity (the free disk will reach its mechanism of plastic collapse at a certain velocity $\bar{\bar{\omega}}$, which may be larger), but, after all, it is almost without any practical sense to discuss the rotation of the disk without connection with the driving shaft.

Of course, the discussion of influence of various factors on the phenomenon of decohesion is very important. The decrease of the plate thickness due to very large negative ε_z (corresponding to very large positive ε_r) will, as a rule, quicken the decohesion. The appearance of the three-dimensional state of stress, connected with "necking", seems to introduce no major changes. However, decohesion may be avoided for certain types of the stress–strain diagrams, not too close to perfect plasticity. On the contrary, if there exists a difference between upper and lower yield point, the phenomenon of decohesion may appear even more frequently.

A special discussion should be devoted to the cases in which the negative strains increase infinitely (due to compression). This case, though resulting also in discontinuities non-permissible from the point of view of continuous media, cannot lead directly to decohesion. It seems, however, that taking here the increase of cross-sectional area into account we simply avoid infinite increase of strains in most cases. The general three-dimensional case requires further investigations.

For the majority of real materials the phenomenon of decohesion should be described by separate limiting equations. But the intention of the present paper is to show that even without such a limitation the discussion of decohesion in perfect plasticity may sometimes be necessary.

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Абстракт—В некоторых задачах идеально упруго-пласти-ческих тел, деформации могут увеличиваться бесконечно, что приводит к разрыве, не допускаемом с точки зрения сплошной среды. Затем, может наступить явление расцепления; соответствующая нагрузка называется "несущей способностью расцепления". Это явление происходит в результате повторного распределения внутренных усилий и как ведет к непосредственному источнению предела несушей способности конструкции или работа конструкции под влиянием приращения параметров нагрузки дальнее всегда возможна. Обсуждаются два соответствующие примеры. Для любого случая, несущая способность расцепления обозначает, с инженерной точки зрения, действительное сопротивление конструкции.